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MATHEMATICS AND COMPUTING MACHINERY SERIES No. 3

JENS G. BALCHEN and ARNE G. BERRE

A Method for Evaluating the Accuracy in the Time Domain Associated with Approximation in the Frequency Domain

Norwegian Contribution No. 3

Trondheim 1959

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A METHOD FOR EVALUATING THE ACCURACY IN THE TIME DOMAIN ASSOCIATED WITH APPROXIMATION IN THE FREQUENCY DOMAIN

by

Jens G. Balchen and Arne G. Berre[x)]

SUMMARY.

A method is outlined which makes it possible by simple means to estimate the error in the time response arising from a certain inaccuracy between an exact and an approximate system in the frequency domain. A new error criterion which makes use of logarithmic amplitude and frequency plots is introduced.

1. INTRODUCTION

One common way to describe the behaviour of linear systems in control and electronic engineering, is by means of frequency response characteristics. These characteristics give the amplitude and phase relationship between a sinusoidal input signal and the resulting (also sinusoidal) output signal. The information one can get from an amplitude-phase-frequency plot will in general be sufficient to characterise the system performance. The frequency response function could be derived either by an analytical procedure or by graphical or experimental methods.

Even though the information contained in the frequency plot gives the full idea about the dynamic behaviour, it is often desirable to be able to indicate the response of a known transient. In the literature there are several methods for determining the transient response when the amplitude-phase-frequency relationship is known [1,2]. Unfortunately most of these methods are complicated and time consuming and restricted to certain classes of input functions. The accuracy associated with these procedures also is difficult to state. A logical procedure, that is often used in practice, is to approximate the actual amplitude-phase-frequency plot with a known and easily synthesized characteristic of such form that the transient could readily be evaluated.

2. THEORY

In this paper it is intended to derive the accuracy necessary for the approximation in the frequency domain to achieve a certain maximum error specified in the time domain.

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Symbols used

$x(t)$, $X(s)$ - input signal, Laplace transform

$y(t)$, $Y(s)$ - output response, Laplace transform

$H(s)$ - transfer function

symbols used without indicies refers to actual system.
Index 1 refers to the approximate system.

$X(j\omega)$, $Y(j\omega)$ etc. is used for $X(s)$, $Y(s)$, for $s=j\omega$ etc.

$|H(j\omega)|$ - amplitude ratio

$\varphi(j\omega)$ - phase shift

$z(t) = y(t) - y_1(t)$ - error in time domain

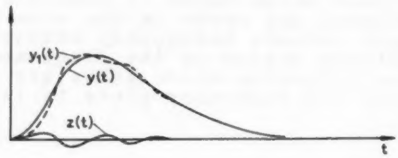


Fig. 1

Fig. 1 shows an example of the three quantities from the last expression above. It is obvious that the approximation is satisfactory when $z(t)$ has a small magnitude. A reasonable measure of the error of the approximation is thus given by the following expression

$$\epsilon = \frac{\int_0^\infty |z(t)|\,dt}{\int_0^\infty |y(t)|\,dt} \tag{1}$$

A small average value of $|z(t)|$ during the settling periode consequently means a good approximation. The integrals in Eq. (1) however, do not easily lend themselves to calculation from the frequency response characteristics only, but it is possible to obtain a good approximation to integrals of the form

$$\int_0^\infty |z(t)|\,dt$$

by using the expression

$$\int_0^\infty z(t)\sin(\omega_1 t + \alpha_1)\,dt \tag{2}$$

and choosing ω_1 and α_1 so as to maximize the integral.

It is easily shown that

$$\int_0^\infty z(t)\sin(\omega_1 t + \alpha_1)dt)_{\max} = |Z(j\omega)|_{\max}$$

$$= |Y(j\omega) - Y_1(j\omega)|_{\max} \qquad (3)$$

This is demonstrated in more detail in ref. [3].
A similar expression can be derived for the denominator in Eq.(1).

Hence

$$\int_0^\infty |y(t)|dt \approx \int_0^\infty y(t)\sin(\omega_2 t + \alpha_2)dt_{\max}$$

$$= |Y(j\omega)|_{\max} \qquad (4)$$

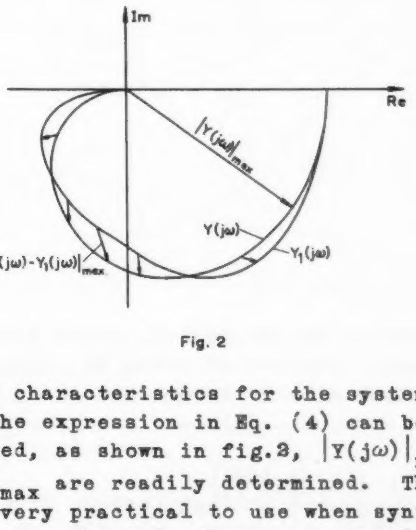


Fig. 2

If the frequency characteristics for the system and the input signal are known, the expression in Eq. (4) can be evaluated. When a polar plot is used, as shown in fig.2, $|Y(j\omega)|_{\max}$ and $|Y(j\omega) - Y_1(j\omega)|_{\max}$ are readily determined. The polar plot, however, is not very practical to use when synthesizing the approximate system. The Bode diagram. using logarithmic scale for amplitude ratio and frequency and linear scale for phase shift. is more flexible for this purpose. Because the difference in Eq. (3) can not be derived directly from separate Bode diagrams of $Y(j\omega)$ and $Y_1(j\omega)$, Eq. (3) is rewritten

$$|Y(j\omega) - Y_1(j\omega)|_{\max} = \left[|Y(j\omega)|\cdot\left|1 - \frac{Y_1}{Y}(j\omega)\right|\right]_{\max}$$

$$= \left[|X(j\omega)|\cdot|H(j\omega)|\cdot\left|1 - \left|\frac{H_1}{H}(j\omega)\right| e^{j(\varphi_1 - \varphi)}\right|\right]_{\max} \qquad (5)$$

The factor

$$|P| = \left| 1 - \left| \frac{H_1}{H}(j\omega) \right| e^{j(\varphi_1 - \varphi)} \right| \qquad (6)$$

can now be determined graphically, when

$$\Delta|H| = \frac{|H_1|}{|H|} \quad \text{and} \quad \Delta\varphi = \varphi_1 - \varphi$$

are given. From the Bode diagram $|H|$ and $|H_1|$ are known in db (decibel) and consequently $\Delta|H|$ is immediately given in db. Fig. 3 shows $|P|$ (db) as a function of $\Delta|H|$ (db) and $\Delta\phi$ (degrees)

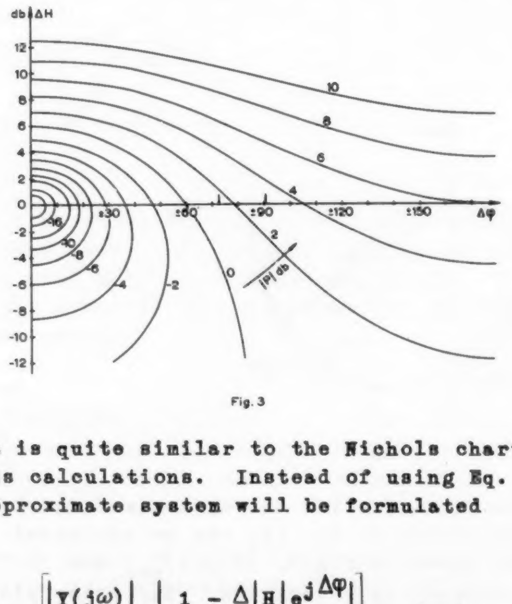


Fig. 3

This plot is quite similar to the Nichols chart used for servo-mechanisms calculations. Instead of using Eq. (1), the accuracy of the approximate system will be formulated

$$\tilde{\epsilon} = \frac{\left[|Y(j\omega)| \; \left| 1 - \Delta|H| e^{j\Delta\varphi} \right| \right]_{max}}{|Y(j\omega)|_{max}} \text{ (db)}$$

$$= \left[|Y| \cdot \left| 1 - \Delta|H| e^{j\Delta\varphi} \right| \right]_{max}\text{(db)} - |Y|_{max}\text{(db)} \qquad (7)$$

Eq. (7) can be interpreted in another way, i.e. that the maximum error $|Y(j\omega) - Y_1(j\omega)|_{max}$ in the frequency domain is a fraction $\tilde{\epsilon}$ of $|Y(j\omega)|_{max}$.

In Eq. (7) $|Y(j\omega)|_{max}$ and $\tilde{\epsilon}$ are supposed to be known. $|Y(j\omega)|_{max}$ is known from the amplitude ratio characteristic and $\tilde{\epsilon}$ is the accuracy that is required in the time domain.

If 1% is specified for the accuracy in the time domain, Eq.(1) will give ϵ = 0,01 and accordingly $\tilde{\epsilon}$ = − 40 db as defined in Eq. (7). From Eq. (6) and Eq. (7) it follows

$$\left[|Y|\cdot|P|\right]_{max}(db) = |Y|_{max}(db) + \tilde{\epsilon}(db) = |Y|_{max}(db)-4o(db) \tag{8}$$

From this equation it is possible to determine corresponding values for $\Delta|H|$ and $\Delta\varphi$ for all frequencies. As an example the following transfer function is chosen

$$H(s) = \frac{T_1 s}{(1+T_1 s)\ (1+T_2 s)}$$

Input signal: $X(s) = \frac{1}{s}$ (step function)

For T_1 = 5 and T_2 = 1 the response y(t) is shown in fig.4.

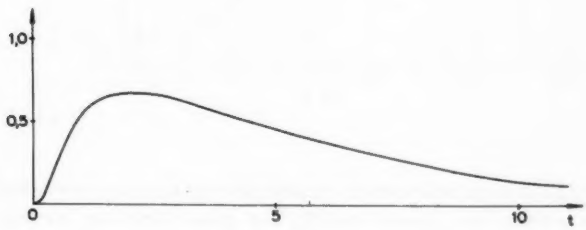


Fig. 4

It is easily shown that

$$|Y(j\omega)|_{max} = T_1 = 5 \simeq 14 \text{ db}$$

The frequency characteristics for the system chosen is plotted in fig. 5. With an $\tilde{\epsilon}$ = −4o db it follows that

$$|P|_{max}(db) = |Y|_{max}(db) + \tilde{\epsilon}(db) - |Y|(db) = -26 - |Y|(db)$$

The figures for $|Y(j\omega)|$ are found in the amplitude characteristic in fig. 5, and by means of fig. 8 the corresponding values of $\Delta|H|$ and $\Delta\varphi$ are determined. In fig. 6 the allowable errors $\Delta|H|$ and $\Delta\varphi$ are shown as functions of frequency. It appears that it is essential to have small deviations for frequencies where $|Y(j\omega)|$ is large in order to maintain accuracy, in this case for low frequencies.

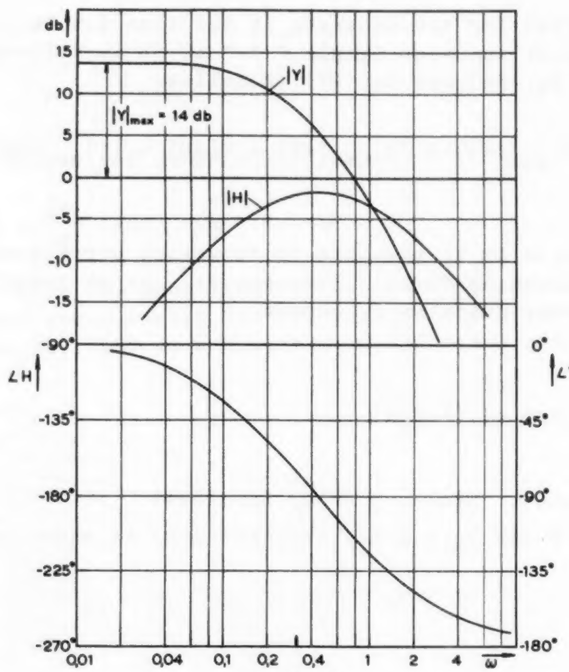


Fig. 5

Further it is discovered that the accuracy is independent of whether the error in phase shift is positive or negative.

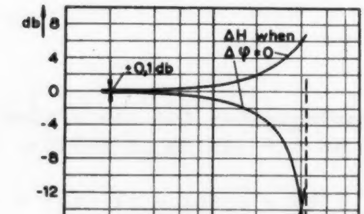


Fig. 6

As to error in amplitude ratio, negative db-values of $\Delta|H|$ are less significant than positive. The preceding discussion also clarifies that the input signal plays an important part in the accuracy resulting from the approximation in the frequency domain.

If the response $y(t)$ is such that

$$\int_0^\infty y(t)\, dt \to \infty$$

which means $|Y(j\omega)|_{max} \to \infty$

there is a need for defining the accuracy in another way than is done in Eq. (1) and Eq. (7). Such a case would arise when the input is a step, $X(s) = \frac{1}{s}$, and

$$\underset{s \to 0}{H(s) \to} \ \text{const} \neq 0$$

It seems then logical to extend the definition given in Eq. (1) to

$$\tilde{\epsilon}' = \frac{\int_0^\infty |s(t)|\, dt}{\int_0^\infty |y(t) - y_\infty(t)|\, dt} \tag{9}$$

where $y_\infty(t)$ is the straight line which $y(t)$ approaches when $t \to \infty$, fig.7.

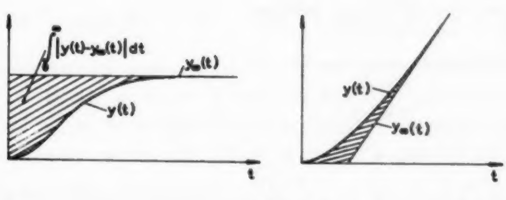


Fig. 7

For the function shown in fig.4. $y_\infty(t) = 0$ and accordingly Eq.(9) leads to Eq. (1). The modified form of Eq. (7) is

$$\tilde{\epsilon}' = \frac{|Y(j\omega) - Y_1(j\omega)|_{max}}{|Y(j\omega) - Y_\infty(j\omega)|_{max}} \tag{10}$$

where $Y_\infty(j\omega)$ is the Laplace transform of $y_\infty(t)$ with $s = j\omega$.

The expression in the denominator in Eq.(10) can be evaluated in the same manner as described in Eq.(5). Doing this, however, requires some additional work and it seems reasonable instead to make an estimation based on the denominator in Eq. (9). If the transient response is known approximately, the area involved should be rather easily determined with a sufficient accuracy.

Another approach is to choose the input function x(t) so that $y_\infty(t) = 0$.

If $X(s) = \frac{1}{s}$

and $H(s) \to \text{const} \neq 0$

X(s) can be modified through the substitution

$$X(s) = \frac{\tau}{1+\tau s}$$

choosing the parameter τ such that

$$\int_0^\infty y(t)\, dt$$

is minimized without the loss of characteristic features associated with the original function.

8. APPLICATION

In order to demonstrate some of the properties of the method outlined in this paper, the system shown in fig. 8. is chosen.

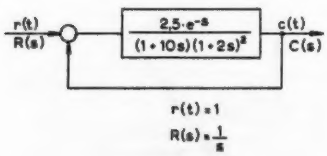


Fig. 8

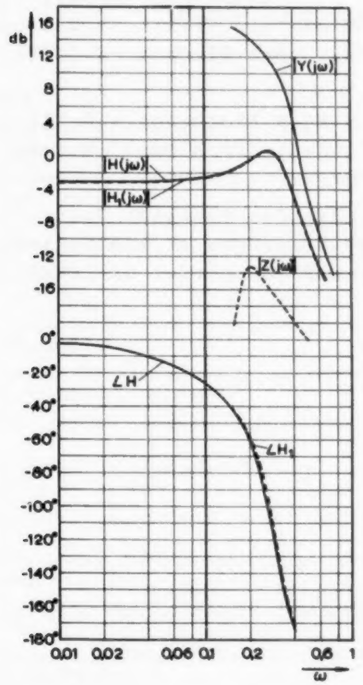
An exact calculation of c(t) for a given r(t) is very complicated. By means of the Nichols chart $\frac{C}{R}(j\omega)$ is evaluated. To approximate c(t), $\frac{C}{R}(j\omega)$ is approximated by functions which more easily leads to the time response. In fig. 9. is shown how $\frac{C}{R}(j\omega)$ is approximated with a second order-system in combination with a transportation lag. The parameters of the approximate system are chosen so that

$$H_1(j\omega) \to H(j\omega)$$

for $j\omega \to 0$

The effect is that

$$\int_0^{\infty}\left[c(t) - c_1(t)\right]dt = 0$$



$$\frac{C}{R}(s) = H(s) = \frac{2,5 \cdot e^{-s}}{(1+10s)(1+2s)^2 + 2,5 \cdot e^{-s}}$$

$$\frac{C}{R}(s)_1 = H_1(s) = \frac{0,71 \cdot (0,32)^2 \cdot e^{-2,11s}}{s^2 + 2 \cdot 0,35 \cdot 0,32s + (0,32)^2}$$

Fig. 9

The $|Z(j\omega)|$ is evaluated by means of fig. 8. and the graphical approach gives a final result

$$|Z(j\omega)|_{max} \approx -18\text{db} = 0,22$$

This value can be compared with an area equivalent to that shaded in fig. 7. With a rise time of about 1,8 sec, an overshoot of about 40 o/o, a transportation lag of 2 sec. and $y_{\infty}(t) = 0,7$, the approximate area is $(0,7 + 0,7 \cdot 0,42)(2 + \frac{1}{2} \cdot 1,8) \approx 2,7$

Hence the accuracy $\frac{0,22}{2,7} \cdot 100 \approx 8$ o/o

In fig. 1o the exact and approximated responses are shown. The solutions were obtained from an analog computer setup.

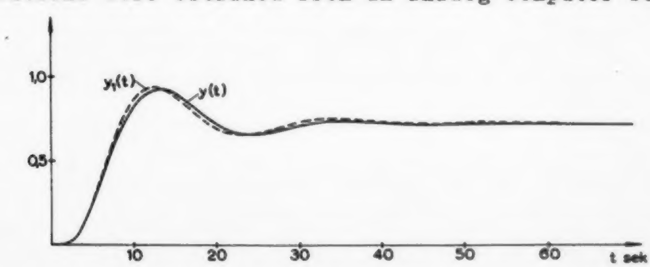


Fig. 10

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